JOURNAL OF APPROXIMATION THEORY 37, 326–334 (1983)

Convex L^p Approximation*

B. L. CHALMERS

Department of Mathematics, University of California, Riverside, California 92521, U.S.A.

AND

A. G. EGGER AND G. D. TAYLOR

Department of Mathematics, Colorado State University, Fort Collins, Colorado 80523, U.S.A.

Communicated by Oved Shisha

Received April 12, 1982

1. INTRODUCTION

In [7] Karlovitz developed an algorithm for finding best L^p approximations from certain finite-dimensional subspaces for p an even integer. It will be shown that this algorithm converges for $2 \leqslant p < \infty$ when the approximating subspace is replaced by certain closed convex subsets of a finite-dimensional subspace. Furthermore, the restrictions which must be placed on the functions involved to ensure convergence are weakened.

We shall consider the problem of approximating 0 by elements of a closed nonempty convex subset K which is contained in a finite-dimensional subspace and which does not contain 0. This is seen to be equivalent to the general problem of approximating a function f by elements of a closed convex subset G, with $f \notin G$ and G contained in a finite-dimensional subspace, by simply translating all functions involved by $-f$. That is,

$$\inf_{v \in G} \|f - v\| = \inf_{u \in G - f} \|u\|$$

and v in G is a best approximation to f from G if and only if $v - f$ is a best approximation from $G - f$ to 0. Finally, since the best approximation is also contained in some sufficiently large ball, we assume henceforth that K is compact rather than closed.

* This research has been supported in part by the National Science Foundation under grant MCS-80-17056.

0021-9045/83 $3.00

For (T, Σ, μ) a positive measure space we define $L^p \equiv L^p(T, \Sigma, \mu)$ to be the Banach space of all μ-equivalence classes of p-summable real-valued functions on T. We define $L^\infty \equiv L^\infty(T, \Sigma, \mu)$ to be the Banach space of real-valued measurable essentially bounded functions on T. For $f \in L^p$, the norm of f, $\|f\|_p$, is defined as usual to be

$$\|f\|_p = \left(\int_T |f|^p \, d\mu\right)^{1/p}, \qquad \text{for} \quad 1 \leqslant p < \infty,$$

$$= \underset{S \in \Sigma, \mu(S)=0}{\text{infimum}} \ \sup_{x \in T \setminus S} |f(x)|, \qquad \text{for} \quad p = \infty.$$

For each $f \in L^p$, the function $\sigma(f(x)) \equiv \text{sign}(f(x))$ is defined by

$$\sigma(f(x)) = +1, \qquad \text{if} \quad f(x) > 0,$$

$$= -1, \qquad \text{if} \quad f(x) < 0,$$

$$= 0, \qquad \text{if} \quad f(x) = 0.$$

We shall require the following well-known results on convexity and differentiability:

THEOREM 1.1 [8, p. 351]. *If (T, Σ, μ) is a measure space, then the norm on $L^p \equiv L^p(T, \Sigma, \mu)$, $1 < p < \infty$, is Gateaux differentiable at each $f \in L^p$, $f \not\equiv 0$. Furthermore, the directional derivative of $\|\cdot\|_p$ at f in the direction $g \in L^p$ is given by*

$$\|\cdot\|_p'(f; g) = \langle g, \|\cdot\|_p' f\rangle = \|f\|_p^{1-p} \int_T |f|^{p-1} \sigma(f) g \, d\mu.$$

In fact, the map $f \to \|\cdot\|_p'(f)$ is continuous at each $f \in L^p, f \not\equiv 0$.

THEOREM 1.2 [5, p. 25]. *Let X be a reflexive Banach space and let F be a convex function defined on a nonempty convex set K contained in X. If F is Gateaux differentiable at each $x \in K$, then F is strictly convex on K if and only if*

$$F(x) > F(y) + \langle F'(y), x - y\rangle$$

for all $x, y \in K$, $x \neq y$.

THEOREM 1.3 [5, p. 37]. *Let X be a Banach space and let F be a convex real-valued Gateaux differentiable function at each point in a convex set K, $K \subset X$. If the map $x \to F'(x)$ is continuous on K and $\hat{x} \in K$, then the following are equivalent.*

$$F(\hat{x}) = \inf_{h \in K} F(h), \tag{1.1a}$$

$$\langle F'(\hat{x}), h - \hat{x} \rangle \geqslant 0 \quad \textit{for all} \quad h \in K, \tag{1.1b}$$

$$\langle F'(h), h - \hat{x} \rangle \geqslant 0 \quad \textit{for all} \quad h \in K. \tag{1.1c}$$

THEOREM 1.4 [8, p. 343]. *The L^p norm, $1 < p < \infty$, is strictly convex on any convex subset of L^p.*

2. EXISTENCE, CHARACTERIZATION, AND UNIQUENESS

Let K be a convex nonempty subset of a Banach space X. If $f \in X \backslash K$ and $g^* \in K$ is such that

$$\|f - g^*\| = \inf_{h \in K} \|f - h\|, \tag{2.1}$$

we shall say g^* is a *best approximation* from K to f.

The following results are well known:

THEOREM 2.1. *If K is a convex closed nonempty subset of L^p such that* $\dim(\operatorname{span}(K)) < \infty$ *and $f \in L^p$, then f has a best approximation g^* from K. If $1 < p < \infty$, then g^* is unique.*

THEOREM 2.2. *If K is a convex closed nonempty subset of $L^p \equiv L^p(T, \Sigma, \mu)$, where (T, Σ, μ) is a measure space and $1 < p < \infty$, and if* $\dim(\operatorname{span}(K)) < \infty$, *then g^* is the best approximation to $f \in L^p$ if and only if*

$$\int_T |f(x) - g^*(x)|^{p-1} \sigma(f(x) - g^*(x))(g^*(x) - g(x))\, d\mu(x) \geqslant 0$$

for all $g \in K$.

For the situation that the convex set K is described via a series of linear constraints, a slightly different appearing characterization theorem can be established. Specifically, let $X = L^p(T, \Sigma, \mu)$, V be an n-dimensional subspace of X, and V' denote its dual in L^q. Let S and W be compact metric spaces. Suppose that functions $G(g, w)$ and $H(g, s)$ from $V \times W$ and $V \times S$ are given such that

(i) for each fixed w, $G(g, w) = \theta_w(g) - \alpha_w$, where θ_w is in V' and $\alpha_w \in \mathbb{R}$ (the reals);

(ii) for each fixed s, $H(g, s) = \psi_s(g) - \beta_s$, where ψ_s is in V' and $\beta_s \in \mathbb{R}$;

(iii) the function $G(g, w)$ is continuous on $V \times W$; $H(g, w)$ is continuous on $V \times S$;

(iv) there exists a $g \in V$ such that $G(g, w) < 0$ for all $w \in W$ and $H(g, s) = 0$ for all $s \in S$.

Given $f \in L^p \backslash V$, with H and G as above, we wish to characterize solutions of the following problem: Define $F(g) = \|f - g\|_p$, $g \in V$, and

$$\text{minimize} \quad F(g) \qquad \text{for} \quad g \in V \tag{P$'$}$$

subject to (a) $G(g, w) \leqslant 0$ for all $w \in W$,

(b) $H(g, s) = 0$ for all $s \in S$.

Theorem 2.1 assures us that there exists a unique solution to problem (P$'$) provided there exists $g \in V$ satisfying (a) and (b) above. (If $g \in V$ satisfies (a) and (b), it is said to be feasible.) In addition, using the theory of [4 or 6] one has

THEOREM 2.3. *Let $X = L^p(T, \Sigma, \mu)$ and let V be an n-dimensional subspace of X. Let G, H, S, and W be as described just prior to problem* (P$'$). *Assume that $f \in X \backslash V$ and that there exists g in V such that $G(g, w) < 0$ for all $w \in W$ and $H(g, s) = 0$ for all $s \in S$. Then $\bar{g} \in V$, $\bar{g}$ feasible, solves problem* (P$'$) *if and only if there exist integers s and s_0 such that $0 \leqslant s_0 \leqslant s \leqslant n$ and such that*

(i) *there are s_0 points $w^k \in \{w \mid G(\bar{g}, w) = 0\}$ for $k = 1,..., s_0$,*

(ii) *there are $s - s_0$ points $s^k \in S$ for $k = s_0 + 1,..., s$, and*

(iii) *there are s real numbers λ_i, $i = 1,..., s$, with $\lambda_i \geqslant 0$ for $1 \leqslant i \leqslant s_0$,*

with the property that on V

$$\nabla_x \|f - \bar{g}\|_p + \sum_{k=1}^{s_0} \lambda_k \nabla_x G(\bar{g}, w^k) + \sum_{k=s_0+1}^{s} \lambda_k \nabla_x H(\bar{g}, s^k) = 0, \tag{2.2}$$

where the identification between $g \in V$ and $x \in \mathbb{R}^n$ is made via the coefficient vector of g with respect to some fixed basis for V.

Equation (2.2) is equivalent to

$$\int_T |f - \bar{g}|^{p-1} \sigma(\bar{g} - f) h \, d\mu + \sum_{k=1}^{s_0} \lambda'_k \theta_{w^k}(h) + \sum_{k=s_0+1}^{s} \lambda'_k \psi_{s^k}(h) = 0$$

for each $h \in V$, where $\lambda'_k \geqslant 0$ for $k = 1,..., s_0$. In this form, Theorem 2.3 is an extension of a result on restricted range approximation in $L^2[a, b]$ due to Levasseur [9]. The form of the constraints problem (P$'$) is originally due to

Chalmers for the L^∞ case [2]. A wide variety of constrained problems such as monotone, positive, and restricted range approximation problems can be stated in the form of problem (P′). The conclusion of Theorem 2.3 can be considered as a "zero in the convex hull of the extreme points" result. For a similar result in L^∞ see [3, p. 73] and for a corresponding L^1 result see [1].

3. The Algorithm

Let (T, Σ, μ) be a finite positive measure space. Let p, $2 \leqslant p < \infty$, be given. Denote by L^p the Banach space $L^p(T, \Sigma, \mu)$. Let K be a compact convex nonempty subset of L^p satisfying

$$0 \notin K, \tag{3.1a}$$

$$\dim(\operatorname{span}(K)) < \infty, \tag{3.1b}$$

$$\mu(\operatorname{supp}(h_1) \cap \operatorname{supp}(h_2)) \neq 0 \qquad \text{for each pair of nonzero elements } h_1 \in K,\ h_2 \in \operatorname{span}(K), \tag{3.1c}$$

and

$$\text{each } h \in K \text{ is also in } L^\infty. \tag{3.1d}$$

Let g^* be the unique best L^p approximate from K to 0. The generalized Karlovitz algorithm for constructing g^* proceeds as follows: Given g_n in K, the algorithm defines two new functions h_n and g_{n+1} in K. First, a new norm $\|\cdot\|_n$ on span(K) is defined by $\|h\|_n = (\int_T |g_n|^{p-2} |h|^2 \, d\mu)^{1/2}$ for $h \in \operatorname{span}(K)$. That $\|\cdot\|_n$ is in fact a weighted L^2 norm on span(K) follows from hypotheses (3.1a)–(3.1d). In addition, $\|\cdot\|_n$ is strictly convex and equivalent to $\|\cdot\|_p$ on span(K). Thus, there exists a unique $h_n \in K$ such that $\|h_n\|_n = \inf_{h \in K} \|h\|_n$. Next, the element $g_{n+1} \in K$ is defined by $g_{n+1} = \lambda g_n + (1 - \lambda) h_n$, where $\lambda \in [0, 1]$ is selected so that $\|g_{n+1}\|_p \leqslant \|\xi g_n + (1 - \xi) h_n\|_p$ for all ξ in $[0, 1]$. Note that λ is unique since L^p is strictly convex. It will be shown in the following that g_n and h_n converge in the p norm to g^*, the best approximation from K to 0.

Remark 1. In the original formulation of this algorithm, Karlovitz [7] assumed that p is an even integer, T is a compact subset of R^n, and μ is Lebesgue measure. Furthermore, K was taken to be a translate of a finite-dimensional subspace V by a function $f \notin V$, where all the functions under consideration are required to be continuous with the added restriction that $\mu(\operatorname{supp}(v - f)) = \mu(T)$ for all $v \in V$. Observe that these hypotheses imply conditions (3.1a)–(3.1d). In particular, for condition (3.1c) we have that if $h_1 = v_0 - f$, $h_2 = \sum_{i=1}^{j} \lambda_i (v_i - f)$ with $v_i \in V$, and $h_1 \not\equiv 0$, then $\mu(\operatorname{supp}(h_1) \cap \operatorname{supp}(h_2)) = 0$ implies that $\mu(\operatorname{supp}(h_2)) = 0$, so that $\sum_{i=1}^{j} \lambda_i v_i - \sum_{i=1}^{j} \lambda_i f = 0$, a.e. Since $f \notin V$, we have $\sum_{i=1}^{j} \lambda_i = 0$ and so $\sum_{i=1}^{j} \lambda_i v_i = 0$ a.e. Hence $h_2 \equiv 0$. Condition (3.1d) holds since T is assumed in this case to be compact. Thus, by considering only a sufficiently large ball in V we have Karlovitz's original algorithm as a special case. (Note also that the algorithm of this paper (and the proof of its convergence, Theorem 3.1 below) shows that the search for λ in $g_{n+1} = \lambda g_n + (1-\lambda) h_n$ can be restricted to $[0, 1]$ rather than $(-\infty, \infty)$ as in [7].)

We now prove that the procedure outlined above results in a convergent algorithm.

THEOREM 3.1. *Let (T, Σ, μ) be a finite positive measure space. Let p, $2 < p \leqslant \infty$, be fixed and let K be a compact convex nonempty subset of L^p satisfying conditions* (3.1a)–(3.1d). *Let $g_0 \in K$ be arbitrary and let the sequences $\{g_n\}$ and $\{h_n\}$ be defined as above. Denote by g^* the unique best L^p approximation from K to* 0. *Then either*

$$\|g_n - g^*\|_p \to 0, \qquad \|g_n\|_p > \|g_{n+1}\|_p, \qquad \|h_n - g^*\|_p \to 0, \tag{3.2a}$$

or

$$\text{there exists } N \text{ such that } g_n \equiv g^* \text{ and } h_n \equiv g^* \text{ for all } n \geqslant N. \tag{3.2b}$$

Proof. We first claim that either $\|g_n\|_p > \|g_{n+1}\|_p$ or else $g_n \equiv g_{n+1}$. Suppose not; then since $\|g_n\|_p \geqslant \|g_{n+1}\|_p$ by construction, we must have that $\|g_n\|_p = \|g_{n+1}\|_p$. Thus, the strict convexity of the L^p norm implies $\|\frac{1}{2}(g_n + g_{n+1})\|_p < \|g_{n+1}\|_p$, which contradicts the minimality of $\|g_{n+1}\|_p$ on the segment $\xi g_n + (1-\xi) h_n$ for $\xi \in [0, 1]$. Thus, $\|g_n\|_p > \|g_{n+1}\|_p$ or $g_n \equiv g_{n+1}$.

Our next assertion is that for $g_n \not\equiv g^*$ either $h_n \equiv g_n$ or $\|g_n\|_p > \|g_{n+1}\|_p$. To establish this we note that $h_n \not\equiv g_n$ implies that $\|h_n\|_n < \|g_n\|_n = \|g_n\|_p$ by the definition of h_n and $\|\cdot\|_n$. Support condition (3.1c) implies that $\|\cdot\|_n$ is strictly convex. Thus, Theorem 1.2 implies that

$$\|h_n\|_n > \|g_n\|_n + \langle \| \ \|_n' g_n, h_n - g_n \rangle,$$

where

$$\langle \| \ \|_n' g_n, h_n - g_n \rangle = \int_T |g_n|^{p-2} g_n (h_n - g_n) \, d\mu$$

by Theorem 1.1. It follows that

$$\int_T |g_n|^{p-1} \sigma(g_n)(h_n - g_n) \, d\mu < 0. \tag{3.3}$$

Now suppose that $\|g_n\|_p = \|g_{n+1}\|_p$. Our previous assertion implies that $g_{n+1} = g_n$ so g_n is $\|\cdot\|_p$ minimal on the segment $\xi g_n + (1-\xi) h_n$, $\xi \in [0, 1]$. Thus, Theorem 1.3 implies that $\langle \| \; \|_p' g_n, h_n - g_n \rangle \geqslant 0$, which is equivalent to $\int_T |g_n|^{p-1} \sigma(g_n)(h_n - g_n)\, d\mu \geqslant 0$, contradicting (3.3). Thus, $\|g_n\|_p > \|g_{n+1}\|_p$ if $h_n \not\equiv g_n$ and $g_n \not\equiv g^*$.

Next, we claim that if $h_n \equiv g_n$, then $g_n \equiv g^*$. Now by Theorem 1.3 h_n is $\|\cdot\|_n$ minimal if and only if

$$\int_T |g_n|^{p-2} h_n(v - h_n)\, d\mu \geqslant 0 \tag{3.4}$$

for all $v \in K$. Thus, if $h_n \equiv g_n$ we have that

$$\int_T |g_n|^{p-2} g_n(v - g_n)\, d\mu \geqslant 0$$

for all $v \in K$, which by Theorem 1.3 applied to $\|\cdot\|_p$ implies $g_n \equiv g^*$ as desired. Thus, either the algorithm terminates in a finite number of steps at g^* or else we have that $\|g_n\|_p > \|g_{n+1}\|_p$ for all n.

Finally, for the completion of (3.2b), we claim that if $g_n = g^*$, then $h_n = g_n$ and $g_m = h_m = g^*$ for all $m \geqslant n$. Indeed, if $g_n = g^*$, then Theorem 1.3 implies that

$$\int_T |g_n|^{p-1} \sigma(g_n)(v - g_n)\, d\mu \geqslant 0$$

for all $v \in K$. Rewriting this as

$$\int_T |g_n|^{p-2} g_n(v - g_n)\, d\mu \geqslant 0$$

for all $v \in K$, we see by Theorem 1.3 applied to $\|\cdot\|_n$ that g_n is minimal from K so that $g_n \equiv h_n$, and hence, $g_{n+1} \equiv g_n \equiv h_n \equiv g^*$, completing the proof of (3.2b). To finish the proof of (3.2a), we observe that if the algorithm does not terminate, then the sequences $\{g_n\}$ and $\{h_n\}$ are contained in K which is a compact set. In addition, our assertions above imply that $\|g_n\|_p > \|g_{n+1}\|_p$ in this case. Now to show that g_n and h_n converge to g^* in $\|\cdot\|_p$, it suffices to show that for each pair of subsequences g_{n_k} and h_{n_k} which converge to g and h, respectively, we have $g^* \equiv g \equiv h$. By (3.1d), g_n and h_n are in L^∞ and the norms $\|\cdot\|_\infty$ and $\|\cdot\|_p$ are equivalent on span(K). Hence, $\|g_{n_k} - g\|_\infty \to 0$ and $\|h_{n_k} - h\|_\infty \to 0$, and in the limit we have

$$\int_T |g|^{p-2} h(v - h)\, d\mu \geqslant 0 \tag{3.5}$$

for all $v \in K$ since (3.4) guarantees

$$\int |g_{n_j}|^{p-2} h_{n_j}(v - h_{n_j})\, d\mu \geqslant 0 \qquad \text{for each} \quad j = 1, 2, \ldots \quad \text{and all } v \in K.$$

Note that if $g \equiv h$, then Theorem 1.3 yields $g \equiv g^*$ since $\int_T |g|^{p-2} g(v-g)\, d\mu \geqslant 0$ for all $v \in K$ implies g is the best L^p approximation to 0 from K. Now if $g \equiv g^*$, then $g \equiv h$ as well by the same argument used to finish the proof of (3.2b). We shall thus assume that $g \not\equiv g^*$ and $g \not\equiv h$ and arrive at a contradiction. If $g \not\equiv g^*$ and $g \not\equiv h$, initialize the algorithm with $g_0 \equiv g$. Then (3.4) and (3.5) imply $h_0 = h$. Since $g \not\equiv h$ and $g \not\equiv g^*$, there must exist $\hat{\lambda}$ in $[0, 1]$ such that $\|\hat{\lambda} g + (1-\hat{\lambda}) h\|_p < \|g\|_p$ by our second assertion in the beginning of this proof. Now $\|g_{n_k}\|_p$ is a strictly decreasing sequence of positive real numbers. Hence, for a sufficiently large k we have that

$$\|g_{n_{k+1}}\|_p \leqslant \|\hat{\lambda} g_{n_k} + (1-\hat{\lambda})\, h_{n_k}\|_p < \|g_{n_{k+1}}\|_p,$$

which is a contradiction. Hence, $g \equiv h$ and $h \equiv g \equiv g^*$ as desired. ∎

Remark 2. In the algorithm, g_{n+1} is not computed directly from g_n but rather is determined by a one-dimensional minimization problem involving g_n and h_n. It is natural to investigate whether the algorithm still converges to g^* if we simplify this procedure by setting $h_n = g_{n+1}$. The following example shows that g_n need not converge in this case.

EXAMPLE. Let (T, Σ, μ) be the interval $[0, 1]$ equipped with Lebesgue measure and Σ be the completion of the Borel sigma algebra on $[0, 1]$. Let $p = 4$. Set $V \equiv \{f \in L^p$ such that $f(x) \equiv r$ for some $r \in \mathbb{R}\}$, $K \equiv \{f \in V$ such that $\|f\|_4 \leqslant 4\}$. We wish to find the best L^4 approximation from K to $\phi(x) = x$. Observe that this is a translation of the problem of Theorem 3.1. In this case the constraint $\|f\|_4 \leqslant 4$ is vacuous. We apply the algorithm as before except that we set $g_{n+1} = h_n$ for each n. Thus, g_{n+1} is determined by

$$\frac{d}{dg} \int_0^1 (x - g_n)^2 (x - g)^2\, dx \,\bigg|_{g_{n+1}} = 0.$$

This leads to a simple recursion formula for g_{n+1},

$$g_{n+1} = \frac{6g_n^2 - 8g_n + 3}{12g_n^2 - 12g_n + 4}. \tag{3.6}$$

Observe that the denominator in (3.6) has no real roots, so that g_{n+1} is well defined. It is easily seen that $g^* = \frac{1}{2}$ by the strict convexity of the L^4 norm. Using (3.6) with $g_0 \equiv 0$, we get the following sequence of iterates:

$$g_0 = 0, \qquad g_1 = \tfrac{3}{4}, \qquad g_2 = \tfrac{3}{14}, \qquad g_3 = \tfrac{153}{194} \cdots.$$

Observe that g_3 is strictly worse, as an L^4 approximation to $\phi(x) = x$, than g_1. This, of course, violates the conclusion of Theorem 3.1. In fact, numerical computations suggest that there exist constants c_1 and c_2 such that $\|g_{2n+1} - c_1\|_4 \to 0$ and $\|g_{2n} - c_2\|_4 \to 0$ as $n \to \infty$ with $c_1 \neq c_2$ and $c_2 = (1 - c_1)$. Solving (3.6) and the equation $1 - g_n = g_{n+1}$, we find that either $g_n = \frac{1}{2}$ or $g_n = (3 \pm \sqrt{3})/6$. Initializing the algorithm with $g_0 = (3 + \sqrt{3})/6$, we may check that $g_{2n} = (3 + \sqrt{3})/6$, $g_{2n+1} = (3 - \sqrt{3})/6$. Thus the algorithm may oscillate if g_{n+1} is chosen to be h_n.

Remark 3. It can be shown that this algorithm converges for $1 < p < 2$ if the zeros of each $g \in K$ are simple.

References

1. B. A. Chalmers, "Convex L^1 Approximation with Emphasis on Approximation by Polynomials with Linear Constraints," preprint of thesis, University of Toronto, Toronto, Ontario, Canada, 1978.
2. B. L. Chalmers, A unified approach to uniform real approximation by polynomials with linear restrictions, *Trans. Amer. Math. Soc.* **166** (1972), 309–316.
3. B. L. Chalmers and G. D. Taylor, Uniform approximation with constraints, *Jahresber. Deutsch. Math.-Verein* **81** (1979), 49–86.
4. E. G. Egger, "Constrained Approximation and Strong Uniqueness in L^p Spaces," Ph.D. Thesis, Colorado State University, Ft. Collins, 1981.
5. I. Ekeland and R. Temam, "Convex Analysis and Variational Problems," North-Holland, Amsterdam, 1976.
6. K. R. Gehner, Characterization theorems for constrained approximation problems via optimization theory, *J. Approx. Theory* **14** (1975), 51–76.
7. L. A. Karlovitz, Construction of nearest points in the L^p, p even, and L^∞ norms, *J. Approx. Theory* **3** (1970), 123–127.
8. G. Koethe, "Topological Vector Spaces I," Springer-Verlag, Berlin, 1969.
9. K. M. Levasseur, "Best Approximation with Respect to Two Objectives," Ph.D. Thesis, University of Rhode Island, Kingston, 1980.
10. O. L. Mangassarian, "Nonlinear Programming," McGraw–Hill, New York, 1969.